Int J Theor Phys (2010) 49: 2306–2311
DOI 10.1007/s10773-010-0418-y

Geometric Phase in the Interaction Between Three-energy Level Atom with New-type Radiation Field

Bao-Lai Zhang · Zhao-Xian Yu

Received: 26 April 2010 / Accepted: 21 June 2010 / Published online: 10 July 2010

Abstract By using of the Lewis-Riesenfeld invariant theory, we have studied the dynamical and the geometric phases in the interaction between three-energy level atom with new-type radiation field, respectively. The Aharonov-Anandan phase is also obtained under the cyclical evolution.

Keywords Geometric phase · Three-energy level atom · Radiation field

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel and Bhandari [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric

B.-L. Zhang (✉)
Dongchang College of Liaocheng University, Liaocheng, Shandong Province 252000, China
e-mail: bao-laizhang@sohu.com

Z.-X. Yu
Department of Physics, Beijing Information Science and Technology University, Beijing 100192, China

phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. In this paper, by using the Lewis-Riesenfeld invariant theory, we shall study the geometric phase in the interaction between three-energy level atom with new-type radiation field.

2 Model

The Hamiltonian in the interaction between three-energy level atom with new-type radiation field can be written as

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \Omega\hat{S}_z + \lambda(t)(\sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\hat{S}_- + \hat{S}_+\hat{a}\sqrt{\hat{a}^\dagger\hat{a}}), \tag{1}$$

where we have considered the case that when the wavelength of the radiation field is larger than the distance between atom and atom). In (1), Ω is the transition frequency of atom, $\lambda(t)$ is the coupling coefficient between atom and radiation field, $\omega(t)$ is the radiation field frequency. $\hat{a}$ and $\hat{a}^\dagger$ are the photon annihilation and creation operators which satisfying relation $[\hat{a}, \hat{a}^\dagger] = 1$. Operators $\hat{S}_\pm$ and $\hat{S}_z$ are

$$\hat{S}_+ = \frac{1}{\sqrt{2}}|3\rangle(\langle 1| + \langle 2|), \quad \hat{S}_- = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)\langle 3|, \tag{2}$$

$$\hat{S}_z = \frac{1}{2}\left[|3\rangle\langle 3| - \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|)\right], \tag{3}$$

and they satisfy commutation relations $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$, $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$.

Introducing $\hat{Q}_+ = \sqrt{\hat{a}^\dagger\hat{a}}\hat{a}^\dagger\hat{S}_-$ and $\hat{Q}_- = \hat{S}_+\hat{a}\sqrt{\hat{a}^\dagger\hat{a}}$, one has

$$\{\hat{Q}_+, \hat{Q}_-\} = \frac{1}{2}\begin{pmatrix} 2(\hat{a}\hat{a}^\dagger)^2 & 0 & 0 \\ 0 & (\hat{a}^\dagger\hat{a})^2 & (\hat{a}^\dagger\hat{a})^2 \\ 0 & (\hat{a}^\dagger\hat{a})^2 & (\hat{a}^\dagger\hat{a})^2 \end{pmatrix} \equiv \hat{\bar{M}}, \tag{4}$$

here $\{,\}$ stands for the anticommuting bracket.

It is easy to find that operators $\hat{\bar{M}}$, $\hat{a}^\dagger\hat{a}$, $\hat{S}_z$, and $\hat{Q}_\pm$ are the supersymmetric generators and form supersymmetric Lie algebra, namely

$$\hat{Q}_-^2 = \hat{Q}_+^2 = 0, \quad [\hat{\bar{M}}, \hat{Q}_\pm] = 0, \quad [\hat{\bar{M}}, \hat{a}^\dagger\hat{a}] = [\hat{S}_z, \hat{a}^\dagger\hat{a}] = [\hat{\bar{M}}, \hat{S}_z] = 0, \tag{5}$$

$$[\hat{Q}_+, \hat{Q}_-] = -2\hat{\bar{M}}\hat{S}_z, \quad \hat{Q}_+ + \hat{Q}_- = -2\hat{S}_z(\hat{Q}_+ - \hat{Q}_-), \quad (\hat{Q}_+ - \hat{Q}_-)^2 = -\hat{\bar{M}}, \tag{6}$$

$$[\hat{Q}_-, \hat{S}_z] = -\hat{Q}_-, \quad [\hat{Q}_+, \hat{S}_z] = \hat{Q}_+, \quad [\hat{Q}_-, \hat{a}^\dagger\hat{a}] = \hat{Q}_-, \quad [\hat{Q}_+, \hat{a}^\dagger\hat{a}] = -\hat{Q}_+. \tag{7}$$

Equation (1) becomes

$$\hat{H} = \omega(t)\hat{a}^\dagger\hat{a} + \Omega\hat{S}_z + \lambda(t)(\hat{Q}_+ + \hat{Q}_-). \tag{8}$$

It is easy to find that $\hat{\bar{M}}|\rangle = 2n^2|\rangle$, where

$$|\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} |n-1\rangle \\ |n\rangle \\ |n\rangle \end{pmatrix}, \tag{9}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra constructed by operators $\hat{\bar{M}}$, $\hat{a}^\dagger\hat{a}$, $\hat{S}_z$, and $\hat{Q}_\pm$. Below, we replace operator $\hat{\bar{M}}$ with the particular eigenvalue $2n^2$.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{\varrho}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial\hat{\varrho}(t)}{\partial t} + [\hat{\varrho}(t), \hat{H}(t)] = 0. \tag{10}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{\varrho}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{11}$$

where $\frac{\partial\lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{12}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (12) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{\varrho}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (12). Then the general solution of the Schrödinger equation (12) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{14}$$

where

$$\delta_n(t) = \int_0^t dt'\langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{15}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (8), we can define the following invariant

$$\hat{\varrho}(t) = \alpha(t)\hat{Q}_+ + \alpha^*(t)\hat{Q}_- + \beta(t)\hat{S}_z. \tag{16}$$

Substituting (8) and (16) into (10), one has the auxiliary equations

$$i\dot{\alpha}(t) + \alpha(t)[\Omega - \omega(t)] - \beta(t)\lambda(t) = 0, \quad i\dot{\beta}(t) + 2\lambda(t)\bar{M}[\alpha^*(t) - \alpha(t)] = 0, \tag{17}$$

where dot denotes the time derivative, and $\bar{M}$ denotes the eigenvalue of operator $\hat{\bar{M}}$.

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t)=\exp[\sigma(t)\hat{Q}_+ - \sigma^*(t)\hat{Q}_-]$. It is easy to find that when satisfying the following relations

$$\sin(2\sqrt{\bar{M}}|\sigma(t)|)=\frac{\sqrt{\bar{M}}[\alpha(t)\sigma^*(t)+\alpha^*(t)\sigma(t)]}{|\sigma(t)|},\quad \beta(t)=\cos(2\sqrt{\bar{M}}|\sigma(t)|), \tag{18}$$

and

$$\begin{aligned}&\frac{\alpha(t)}{2}[1+\cos(2\sqrt{\bar{M}}|\sigma(t)|)]-\frac{\beta(t)\sigma(t)}{2\sqrt{\bar{M}}|\sigma(t)|}\sin(2\sqrt{\bar{M}}|\sigma(t)|)\\&\quad-\frac{\alpha^*(t)\sigma^{*2}(t)}{2|\sigma(t)|^2}[1-\cos(2\sqrt{\bar{M}}|\sigma(t)|)]=0,\end{aligned} \tag{19}$$

then a time-independent invariant appears

$$\hat{\varrho}_V \equiv \hat{V}^\dagger(t)\hat{\varrho}(t)\hat{V}(t)=\hat{S}_z. \tag{20}$$

From (18), we can let

$$\sigma(t)=\frac{\theta(t)}{2\sqrt{\bar{M}}}\exp[i\gamma(t)],\quad \alpha(t)=\frac{\sin\theta(t)}{2\sqrt{\bar{M}}}\exp[i\gamma(t)],\quad \theta(t)=2\sqrt{\bar{M}}|\sigma(t)|. \tag{21}$$

From (21), the invariant $\hat{I}(t)$ in (16) becomes

$$\hat{\varrho}(t)=\frac{\sin\theta(t)}{2\sqrt{\bar{M}}}\{\exp[i\gamma(t)]\hat{Q}_+ + \exp[-i\gamma(t)]\hat{Q}_-\}+\cos\theta(t)\hat{S}_z. \tag{22}$$

By using of the Baker-Campbell-Hausdoff formula [28]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots, \tag{23}$$

with $\hat{V}(t)=\exp[\hat{L}(t)]$, it is easy to find that when satisfying the following equation

$$[\omega(t)-\Omega]\sin\theta(t)+2\lambda(t)\sqrt{\bar{M}}[\cos\theta(t)\cos\gamma(t)-i\sin\gamma(t)]-i\dot{\theta}(t)+\dot{\gamma}(t)\sin\theta(t)=0, \tag{24}$$

one has

$$\begin{aligned}\hat{H}_V(t)&=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}\\&=\omega(t)\hat{a}^\dagger\hat{a}+\{\omega(t)[1-\cos\theta(t)]+\Omega\cos\theta(t)+2\lambda(t)\sqrt{\bar{M}}\sin\theta(t)\cos\gamma(t)\}\hat{S}_z\\&\quad+\dot{\gamma}(t)[1-\cos\theta(t)]\hat{S}_z.\end{aligned} \tag{25}$$

We can obtain the particular solutions of the time-dependent Schrödinger equation (12),

$$|\psi_{S_z}(t)\rangle=\exp\left\{-i\int_0^t[\dot{\delta}^d_{S_z}(t')+\dot{\delta}^g_{S_z}(t')]dt'\right\}\hat{V}(t')|\rangle\quad (S_z=0,\pm1), \tag{26}$$

where

$$\dot{\delta}^d_{S_z=+1}(t') = (n-1)\omega(t') + \omega(t')[1-\cos\theta(t')] + \Omega\cos\theta(t') + 2\sqrt{2}n\lambda(t')\sin\theta(t')\cos\gamma(t'), \tag{27}$$

$$\dot{\delta}^g_{S_z=+1}(t') = \dot{\gamma}(t')[1-\cos\theta(t')], \tag{28}$$

$$\dot{\delta}^d_{S_z=-1}(t') = n\omega(t') - \omega(t')[1-\cos\theta(t')] - \Omega\cos\theta(t') - 2\sqrt{2}n\lambda(t')\sin\theta(t')\cos\gamma(t'), \tag{29}$$

$$\dot{\delta}^g_{S_z=-1}(t') = -\dot{\gamma}(t')[1-\cos\theta(t')], \tag{30}$$

$$\dot{\delta}^d_{S_z=0}(t') = n\omega(t'), \quad \dot{\delta}^g_{S_z=0}(t') = 0. \tag{31}$$

From (27)–(31), we conclude that the dynamical and the geometric phases of the system are $\exp[-i\int_0^t \dot{\delta}^d_{S_z}(t')dt']$ and $\exp[-i\int_0^t \dot{\delta}^g_{S_z}(t')dt']$ with $S_z = 0, \pm 1$, receptively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{\varrho}(t)$ and let $\theta(t)$=constant, one has from (28), (30) and (31)

$$\delta^g_{S_z}(T) = \begin{cases} 2\pi(1-\cos\theta), & (S_z=+1), \\ -2\pi(1-\cos\theta), & (S_z=-1), \\ 0, & (S_z=0). \end{cases} \tag{32}$$

Here $2\pi(1-\cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{\varrho}(t)$, (32) is the known Aharonov-Anandan phase.

References

1. Pancharatnam, S.: Proc. Indian Acad. Sci., Sect. A Phys. Sci. **44**, 247 (1956)
2. Berry, M.V.: Proc. R. Soc. Lond., A Math. Phys. Sci. **392**, 45 (1984)
3. Aharonov, Y., Anandan, J.: Phys. Rev. Lett. **58**, 1593 (1987)
4. Samuel, J., Bhandari, R.: Phys. Rev. Lett. **60**, 2339 (1988)
5. Mukunda, N., Simon, R.: Ann. Phys. (N.Y.) **228**, 205 (1993)
6. Pati, A.K.: Phys. Rev. A **52**, 2576 (1995)
7. Uhlmann, A.: Rep. Math. Phys. **24**, 229 (1986)
8. Sjöqvist, E.: Phys. Rev. Lett. **85**, 2845 (2000)
9. Tong, D.M., et al.: Phys. Rev. Lett. **93**, 080405 (2004)
10. Lewis, H.R., Riesenfeld, W.B.: J. Math. Phys. **10**, 1458 (1969)
11. Gao, X.C., Xu, J.B., Qian, T.Z.: Phys. Rev. A **44**, 7016 (1991)
12. Gao, X.C., Fu, J., Shen, J.Q.: Eur. Phys. J. C **13**, 527 (2000)
13. Gao, X.C., Gao, J., Qian, T.Z., Xu, J.B.: Phys. Rev. D **53**, 4374 (1996)
14. Shen, J.Q., Zhu, H.Y.: arXiv:quant-ph/0305057v2 (2003)
15. Richardson, D.J., et al.: Phys. Rev. Lett. **61**, 2030 (1988)
16. Wilczek, F., Zee, A.: Phys. Rev. Lett. **25**, 2111 (1984)
17. Moody, J., et al.: Phys. Rev. Lett. **56**, 893 (1986)
18. Sun, C.P.: Phys. Rev. D **41**, 1349 (1990)
19. Sun, C.P.: Phys. Rev. A **48**, 393 (1993)
20. Sun, C.P.: Phys. Rev. D **38**, 298 (1988)
21. Sun, C.P., et al.: J. Phys. A **21**, 1595 (1988)
22. Sun, C.P., et al.: Phys. Rev. A **63**, 012111 (2001)
23. Chen, G., Li, J.Q., Liang, J.Q.: Phys. Rev. A **74**, 054101 (2006)

24. Chen, Z.D., Liang, J.Q., Shen, S.Q., Xie, W.F.: Phys. Rev. A **69**, 023611 (2004)
25. He, P.B., Sun, Q., Li, P., Shen, S.Q., Liu, W.M.: Phys. Rev. A **76**, 043618 (2007)
26. Li, Z.D., Li, Q.Y., Li, L., Liu, W.M.: Phys. Rev. E **76**, 026605 (2007)
27. Niu, Q., Wang, X.D., Kleinman, L., Liu, W.M., Nicholson, D.M.C., Stocks, G.M.: Phys. Rev. Lett. **83**, 207 (1999)
28. Wei, J., Norman, E.: J. Math. Phys. **4**, 575 (1963)